

Universality and constant scalar curvature invariants

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Abstract

A classical solution is called universal if the quantum correction is a multiple of the metric. Universal solutions consequently play an important role in the quantum theory. We show that in a spacetime which is universal all of the scalar curvature invariants are constant (i.e., the spacetime is *CSI*).

1 Universality

In [1] metrics of holonomy $\text{Sim}(n-2)$ were investigated, and it was found that all 4-dimensional $\text{Sim}(2)$ metrics (which belong to the subclass of Kundt-*CSI* spacetimes [2]) are universal and consequently can be interpreted as metrics with vanishing quantum corrections and are automatically solutions to the quantum theory.

A classical solution is called *universal* if the quantum correction is a multiple of the metric, and therefore plays an important role in the quantum theory regardless of what the exact form of this theory might be. That is, if the spacetime is universal, then every symmetric conserved rank-2 tensor, T_{ab} , which is constructed from the metric, Riemann tensor and its covariant derivatives, is of the form

$$T_{ab} = \mu g_{ab}, \quad (1)$$

where μ is a constant. Now, for every scalar S that appears in the action (gravitational Lagrangian) we obtain by variation (since these geometric tensors are automatically conserved due to the invariance of the actions under spacetime diffeomorphisms) a symmetric conserved rank-2 tensor S_{ab} . For each such tensor we have from the condition of universality that $S_{ab} = \hat{\mu} g_{ab}$. By using an

appropriate set of such scalars, we shall show that all of the scalar curvature invariants are constant and that the resulting spacetimes are therefore CSI (by definition). Since the resulting spacetime is automatically an Einstein space, in effect we must show that all scalar contractions of the Weyl tensor and its derivatives are constants [3, 4]. We utilize the results of FKWC [5, 6] to obtain all conserved rank-2 tensors obtained from variations (by Noether's theorem) of an elemental scalar Riemann polynomial.

There are a number of related results we would like to investigate in this paper. We will state these in terms of a conjecture and will corroborate this conjecture by proving a number of sub-results using a number of different arguments.

Conjecture 1.1. *A Universal n -dimensional Lorentzian spacetime, (M, g) , has the following properties:*

1. *It is CSI.*
2. *It is a degenerate Kundt spacetime.*
3. *There exists a spacetime, (\tilde{M}, \tilde{g}) , of Riemann type **D** having identical scalar polynomial invariants; consequently (\tilde{M}, \tilde{g}) is spacetime homogeneous.*
4. *There exists a homogeneous isotropy-irreducible Riemannian spacetime (\hat{M}, \hat{g}) having identical scalar polynomial invariants as (M, g) ; i.e., (\hat{M}, \hat{g}) is universal as a Riemannian space.*

In low dimensions this conjecture can be proven; in particular, dimension 2 is trivial as there is only one independent component, namely the Ricci scalar R . In dimension 3, there are only Ricci invariants and the conjecture can be proven by brute force using symmetric conserved tensors. Most of our investigation will focus on dimension 4 and, unless stated otherwise, hereafter we will assume that the manifold is 4 dimensional.

2 The CSI result

Let us first present results substantiating the claim that universal spacetimes are CSI. This is clearly the case in the Riemannian case where Bleecker [7] showed that the critical manifolds are homogeneous and, hence, CSI. Note that in the Riemannian case a CSI space is equivalent to a locally homogeneous space; however, in the Lorentzian case these are not equivalent as there are many examples of CSI spacetimes not being locally homogeneous.

2.1 The direct method

Field theoretic calculations on curved spacetimes are non-trivial due to the systematic occurrence, in the expressions involved, of Riemann polynomials. These polynomials are formed from the Riemann tensor by covariant differentiation, multiplication and contraction. The results of these calculations are complicated because of the non-uniqueness of their final forms, since the symmetries of the Riemann tensor as well as the Bianchi identities can not be used in a uniform manner and monomials formed from the Riemann tensor may be linearly

dependent in non-trivial ways. In [5], Fulling, King, Wybourne and Cummings (FKWC) systematically expanded the Riemann polynomials encountered in calculations on standard bases constructed from group theoretical considerations. They displayed such bases for scalar Riemann polynomials of order eight or less in the derivatives of the metric tensor and for tensorial Riemann polynomials of order six or less. We adopt the FKWC-notations $\mathcal{R}_{s,q}^r$ and $\mathcal{R}_{\{\lambda_1 \dots\}}^r$ to denote, respectively, the space of Riemann polynomials of rank r (number of free indices), order s (number of differentiations of the metric tensor) and degree q (number of factors $\nabla^p R_{\dots}$) and the space of Riemann polynomials of rank r spanned by contractions of products of the type $\nabla^{\lambda_1} R_{\dots}$ [5]. The geometrical identities utilized to eliminate “spurious” Riemann monomials include: (i) the commutation of covariant derivatives, (ii) the “symmetry” properties of the Ricci and the Riemann tensors (pair symmetry, antisymmetry, cyclic symmetry), and (iii) the Bianchi identity and its consequences obtained by contraction of index pairs.

In this paper we actually use a slightly modified version of the FKWC-bases [6], which are independent of the dimension of spacetime and provide irreducible expressions for all of our results. In addition, the results of [6] provide irreducible expressions for the metric variations (i.e., for the functional derivatives with respect to the metric tensor) of the action terms associated with the 17 basis elements for the scalar Riemann polynomials of order six in derivatives of the metric tensor (the so-called curvature invariants of order six).

2.1.1 Riemann polynomials of rank 0 (scalars)

The most general expression for a scalar of order six or less in derivatives of the metric tensor is obtained by expanding it in the FKWC-basis for Riemann polynomials of rank 0 and order 6 or less [5].

The sub-basis for Riemann polynomials of rank 0 and order 2 consists of a single element: R [$\mathcal{R}_{2,1}^0$].

Choosing S to be the Ricci scalar, R , we find that the Einstein tensor is conserved and $R_{ab} = \lambda g_{ab}$, where λ is a constant, and the spacetime is necessarily an Einstein space:

$$R_{pq} = \lambda g_{pq}; \quad R_{pq;r} = 0. \quad (2)$$

Every scalar contraction of the Ricci tensor (or its covariant derivatives, which are in fact zero), will thus necessarily be constant. Every scalar contraction of the Riemann tensor and its derivatives with the Ricci tensor or its covariant derivatives will be constant. For example, for $S = R_{ab}R^{ab}$ for an Einstein space we have that $S_{ab} = 2(R_{acbd} - \frac{1}{4}g_{ab}R_{cd})R^{cd} = \tilde{\mu}g_{ab}$ (where $\tilde{\mu} \sim \lambda^2$). Every mixed invariant (containing both the Ricci tensor and the Weyl tensor and their derivatives, will be constant or can be written entirely as a contraction of scalars involving just the Weyl tensor and its derivatives (up to an additive constant term).

Thus to prove that the resulting spacetimes are CSI, we must show that all scalar contractions of the Weyl tensor and its derivatives are constants.

The sub-basis for Riemann polynomials of rank 0 and order 4 has 4 elements: $\square R$ [$\mathcal{R}_{4,1}^0$]: R^2 , $R_{pq}R^{pq}$, $R_{pqrs}R^{pqrs}$ [$\mathcal{R}_{4,2}^0$].

From (2) there is only one rank 0/order 4 independent scalar, $C^2 \equiv C_{pqrs}C^{pqrs}$. By varying $S = C^2$, we obtain a symmetric conserved rank-2 tensor which

depends on quadratic polynomial contractions of the Weyl tensor, which by universality is proportional to the metric:

$$C_{almn}C_b^{lmn} + C_{blmn}C_a^{lmn} = 2\hat{\lambda}g_{ab}. \quad (3)$$

Hence we have that:

$$C^2 = \hat{\lambda}. \quad (4)$$

Indeed, by choosing S to be a polynomial contraction of the Weyl tensor alone (higher than quadratic), we find that by varying S we obtain symmetric conserved rank-2 tensors which depends on polynomial contractions of the Weyl tensor which by universality are proportional to the metric, and hence all zeroth order invariants constructed from the Weyl tensor are constant (and the spacetime is said to be CSI₀). We note that in higher dimensions, all Lovelock tensors are divergence free and consequently (by universality) proportional to the metric. However, we shall not proceed in this way here.

The most general expression for a gravitational Lagrangian of order six in derivatives of the metric tensor is obtained by expanding it in the FKWC-basis for *Riemann polynomials of order 6 and rank 0*. This sub-basis consists of the 17 following elements [5]: $\square\square R$ [$\mathcal{R}_{6,1}^0$]: $R\square R$, $R_{;pq}R^{pq}$, $R_{pq}\square R^{pq}$, $R_{pq;rs}R^{pqrs}$ [$\mathcal{R}_{\{2,0\}}^0$]: $R_{;p}R^p$, $R_{pq;r}R^{pq;r}$, $R_{pq;r}R^{pr;q}$, $R_{pqrs;t}R^{pqrs;t}$ [$\mathcal{R}_{\{1,1\}}^0$]: R^3 , $RR_{pq}R^{pq}$, $R_{pq}R^p_r R^{qr}$, $R_{pq}R_{rs}R^{pqrs}$, $RR_{pqrs}R^{pqrs}$, $R_{pq}R^p_{rst}R^{qrst}$, $R_{pqrs}R^{pquv}$, R^{rs}_{uv} , $R_{pqrs}R^p_u R^q_v R^{rusv}$ [$\mathcal{R}_{6,3}^0$].

In general, only 10 of these give rise to independent variations. The other 7 depend on these via total divergences (and Stokes theorem); the functional derivatives (i.e., conserved tensors) with respect to the metric tensor of the 7 remaining action terms can then be obtained in a straightforward manner.

In the case of an Einstein space satisfying the conditions (2), (3) and (4), there are only three independent rank 0/order 6 scalars:

$$(\nabla C)^2 \equiv C_{pqrs;t}C^{pqrs;t}, C_1^3 \equiv C_{pqrs}C^{pquv}C^{rs}_{uv}, C_2^3 \equiv C_{pqrs}C^p_u C^q_v C^{rusv} \quad (5)$$

(where, for example, $(\nabla C)^2 \equiv R_{pqrs;t}R^{pqrs;t} = C_{pqrs;t}C^{pqrs;t}$).

Variations of the last four scalars in the list above give rise to 4 independent conserved rank-2 tensors (although $RR_{pqrs}R^{pqrs}$ and $R_{pq}R^p_{rst}R^{qrst}$ are equivalent to $\lambda\hat{\lambda}$, their variations are non-trivial). Note that $R_{pqrs;t}R^{pqrs;t}$ depends on the other 4 scalars via a total divergence (and Stokes theorem).

2.1.2 Conserved rank 2 tensors of order six

The functional derivatives of the ten independent action terms in the FKWC-basis were expanded in [6]: for an Einstein space satisfying the conditions (2), (3) and (4), we obtain the following 4 independent explicit irreducible expressions for the metric variations of the action terms constructed from the 17 scalar Riemann monomials of order six:

$$\begin{aligned} H_{ab}^{(6,3)(7)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pqrs} R^{pquv} R^s_{uv} \\ &= 24 R^p_{(a} R^{qr}_{|pqr|b)} - 12 R^p_{a;q} R^q_{pb} + 12 R^p_{a;q} R^q_{b;p} \\ &\quad + 3 R^{pqrs}_{;a} R_{pqrs;b} - 6 R^{pqr}_{a;s} R^{rs}_{pqb} - 6 R^{pq} R^s_{pa} R_{rsqb} \\ &\quad + 12 R^{pqrs} R^t_{pqa} R_{trsb} + \frac{1}{2} g_{ab} [R_{pqrs} R^{pquv} R^s_{uv}], \end{aligned} \quad (6)$$

which implies that (using (2), (3)-(4))

$$3 C^{pqrs}{}_{;a} C_{pqrs;b} - 6 C^{pqr}{}_{a;s} C_{pqr}{}^{;s} + 12 C^{prqs} C^t{}_{pqa} C_{trsb} + \frac{1}{2} g_{ab} [C_{pqrs} C^{pquv} C^{rs}{}_{uv}] = \lambda_1 g_{ab}. \quad (7)$$

In addition,

$$\begin{aligned} H_{ab}^{(6,3)(5)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R R_{pqrs} R^{pqrs} \\ H_{ab}^{(6,3)(6)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pq} R^p{}_{rst} R^{qrst} \\ H_{ab}^{(6,3)(8)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{prqs} R^p{}^q{}_u R^{rusv} \end{aligned} \quad (8)$$

yield (respectively),

$$2 C^{pqrs} C_{pqrs;(ab)} + 2 C^{pqrs}{}_{;a} C_{pqrs;b} + g_{ab} [-2 C_{pqrs;t} C^{pqrs;t} + 2 C_{pqrs} C^{pquv} C^{rs}{}_{uv} + 8 C_{prqs} C^p{}^q{}_u C^{rusv}] = \lambda_2 g_{ab}, \quad (9)$$

$$\begin{aligned} &\frac{1}{2} C^{pqrs} C_{pqrs;(ab)} + \frac{1}{2} C^{pqrs}{}_{;a} C_{pqrs;b} - C^{pqr}{}_{a;s} C_{pqr}{}^{;s} \\ &+ C^{pqrs} C_{pqta} C_{rs}{}^t{}_b + 4 C^{prqs} C^t{}_{pqa} C_{trsb} - C^{pqr}{}_s C_{pqrt} C^s{}^t{}_a \\ &+ \frac{1}{4} g_{ab} [-C_{pqrs;t} C^{pqrs;t} + C_{pqrs} C^{pquv} C^{rs}{}_{uv} + 4 C_{prqs} C^p{}^q{}_u C^{rusv}] = \lambda_3 g_{ab}, \quad (10) \end{aligned}$$

$$\begin{aligned} &-\frac{3}{4} C^{pqrs} C_{pqrs;(ab)} + \frac{3}{4} C^{pqrs}{}_{;a} C_{pqrs;b} - \frac{3}{2} C^{pqrs} C_{pqta} C_{rs}{}^t{}_b - 9 C^{prqs} C^t{}_{pqa} C_{trsb} \\ &+ \frac{3}{2} C^{pqr}{}_s C_{pqrt} C^s{}^t{}_a + \frac{1}{2} g_{ab} [C_{prqs} C^p{}^q{}_u C^{rusv}] = \lambda_4 g_{ab}. \quad (11) \end{aligned}$$

Contracting eqns. (7), (9)-(11), and using $C^{pqrs} \square C_{pqrs} = -C_1^3 - 4C_2^3 + 2\lambda\hat{\lambda}$ (etc.) [6], we then obtain

$$\begin{aligned} -3(\nabla C)^2 + 2C_1^3 + 12C_2^3 &= 4\lambda_1, \\ -3(\nabla C)^2 + 3C_1^3 + 4C_2^3 &= 2\lambda_2 - 2\lambda\hat{\lambda}, \\ -3(\nabla C)^2 + 3C_1^3 + 12C_2^3 &= 8\lambda_3 - 2\lambda\hat{\lambda}, \\ -3(\nabla C)^2 + 3C_1^3 + 16C_2^3 &= -16\lambda_4 - 6\lambda\hat{\lambda}, \end{aligned} \quad (12)$$

and hence the 3 independent scalars of order 6 are all constant:

$$(\nabla C)^2 = \mu_1, \quad C_1^3 = \mu_2, \quad C_2^3 = \mu_3. \quad (13)$$

Since all of the basis scalars of order six are constant, then all scalars of order six are constant.

We now proceed with the higher order scalars: orders (8,10,12) were considered in [5]. In particular, there is a sub-basis of scalar (rank-0) order 8

polynomials consisting of 92 elements given in Appendix B of [5] from which, by variation, we can obtain a set of independent conserved rank-2 tensors of order 8. For an Einstein space satisfying (2), (3)-(4) and (13), there are only 11 independent scalars: $C^{pqrs;tu}C_{pqrs;tu}$ and $C^{pqrs}C_p^{tuv}C_{qtru;sv}$, 3 scalars (involving squares of the first covariant derivative) of the form $C^{pqrs}C^{tuv}_{p;q}C_{tuwr;s}$, and 6 algebraic fourth order polynomials of the form $C^{pqrs}C_{pqr}^t C^{uvw}_s C_{uvwt}$. By obtaining the set of (more than 12) independent conserved rank-2 tensors of order 8, it follows that all of these 11 independent scalars are constant. In particular, the 3 scalars involving the first covariant derivative of the Weyl tensor are constant, and we are well on our way to showing that the spacetime is CSI_1 . Indeed, in four-dimensions this is sufficient to show that the resulting spacetime is CSI [3, 4]. Continuing in this way we obtain the result that in a universal spacetime all scalar curvature invariants are constant.

An alternative proof, at least in a restricted range of applicability, is provided by the slice theorem.

2.2 The slice theorem

Let I_i denote all possible polynomial scalar curvature invariants. Then we can generate a corresponding set of conserved symmetric tensors, $T_{i,\mu\nu}$, by considering the variation of $S[I_i] = \int I_i \sqrt{-g} d^N x$.

Let us assume that the spacetime under consideration is universal. If the spacetime is strongly universal then all of these symmetric tensors are zero: $T_{i,\mu\nu} = 0$. If there is a $T_{i,\mu\nu}$ which is non-zero, then the spacetime is weakly universal, and, assuming that $T_{1,\mu\nu} = \lambda_1 g_{\mu\nu} \neq 0$, we can then define the equivalent set of invariants:

$$\tilde{I}_1 = I_1 + 2\lambda_1, \quad \tilde{I}_i = I_i - \frac{\lambda_i}{\lambda_1} I_1.$$

We notice that for this new set of invariants, the corresponding conserved tensors are all zero: $\tilde{T}_{i,\mu\nu} = 0$.

This means that we have a full set of invariants each of which has a zero variation: $\frac{\delta S}{\delta g_{\mu\nu}} = 0$. This is a signal that a universal metric has a degeneracy in its curvature structure. In particular, consider a metric variation $\delta g_{\mu\nu} = \epsilon h_{\mu\nu}$, where $h_{\mu\nu} g^{\mu\nu} = 0$ (traceless). This implies that the variation with respect to the metric is zero, which then implies that the variation of all the invariants in the direction of $h_{\mu\nu}$ is zero. The metric is thus a fixed point of all possible actions.

For the degenerate Kundt metrics there exists a one-parameter family of metrics g_τ such that $I_i[g] = I_i[g_\tau]$. Clearly, this implies that $\lim_{\tau \rightarrow 0} \frac{S[g] - S[g_\tau]}{\tau} = 0$; i.e., $\frac{\delta S}{\delta g_{\mu\nu}}$ vanishes along $h_{\mu\nu} \equiv \lim_{\tau \rightarrow 0} (g_{\mu\nu} - g_{\tau,\mu\nu})/\tau$. We note that for the Kundt spacetimes this metric deformation can always be chosen to be traceless (indeed, nilpotent). The universality condition leads to additional conditions since *all* variations of the metric is required to be zero. However, degenerate Kundt metrics are particularly promising candidates for universal metrics [8].

In the Riemannian case, the slice theorem was used by Bleeker [7] to prove many results regarding critical metrics. The slice theorem considers the manifold of metrics modulo the diffeomorphism group. Ebin [9] proved the slice theorem for the compact Riemannian case. The Lorentzian case is more problematic and its general validity is questionable, but Isenberg and Marsden [10]

showed a slice theorem for solutions to the Einstein equations given some assumptions (essentially, global hyperbolicity and compact spatial sections). In its infinitesimal version, it states that any symmetric tensor can be split as follows:

$$S_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} + T_{\mu\nu}, \quad (14)$$

for some vector field, X , and where T is conserved: $\nabla^\mu T_{\mu\nu} = 0$. The vector field X can be interpreted as the generator of the diffeomorphism group and thus corresponds to a “gauge freedom”.

Consider the Lorentzian case when the slice theorem is valid. It can now be shown that universality implies CSI (following parts of Bleecker’s argument). Assume therefore that the spacetime is *not* CSI. Then there must exist a non-constant invariant I . In particular, there must exist a non-trivial interval $[a, b]$ onto which the invariant I is onto. Therefore, choose a sufficiently small interval and a smooth function $f(I)$. The space of such functions is clearly infinite dimensional. Construct then the tensor deformation $\tilde{g}_{\mu\nu} = (1 + f(I))g_{\mu\nu}$. By the slice theorem, there exists a diffeomorphism ϕ such that $\phi^* \tilde{g}_{\mu\nu}$ is conserved. Clearly, $\phi^* \tilde{g}_{\mu\nu}$ is an invariant tensor and thus, by universality, $\phi^* \tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$. This implies that the metric deformation is a conformal transformation. However, the space of conformal transformations is finite, thus, it must be possible to choose a $f(I)$ such that $\phi^* \tilde{g}_{\mu\nu} \neq \lambda g_{\mu\nu}$. Consequently, the space is not universal. To summarise, if a spacetime is not CSI, then it is not universal. Therefore, *universality implies CSI*.

Note that in the compact Riemannian case the slice theorem holds and thus universality implies CSI. In the Riemannian case CSI implies local homogeneity and thus this provides us with a slightly different proof to that of Bleecker [7].

Of course, this result depends crucially on the range of applicability of the slice theorem. It is consequently of interest to determine for which Lorentzian spaces the slice theorem is valid. However, the result is important in the context here, since it can be seen that there is a clear link between universality and CSI spaces, which lends further support to the conjecture.

Let us next consider some properties of the Kundt-CSI spacetimes.

3 Kundt CSI metrics

In [4] it was proven that if a 4D spacetime is *CSI*, then either the spacetime is locally homogeneous or the spacetime is a degenerate Kundt spacetime. The Kundt-*CSI* spacetimes are of particular interest since they are solutions of supergravity or superstring theory when supported by appropriate bosonic fields [11]. It is plausible that a wide class of *CSI* solutions are exact solutions to string theory non-perturbatively [12]. In the context of string theory, it is of considerable interest to study higher dimensional Lorentzian *CSI* spacetimes. In particular, a number of higher-dimensional *CSI* spacetimes are also known to be solutions of supergravity theory [11]. The supersymmetric properties of *CSI* spacetimes have also been studied, particularly those that admit a null covariantly constant vector (CCNV) [13].

A Kundt-*CSI* can be written in the form [2]

$$ds^2 = 2du [dv + H(v, u, x^k)du + W_i(v, u, x^k)dx^i] + g_{ij}^\perp(x^k)dx^i dx^j, \quad (15)$$

where the metric functions H and W_i , requiring CSI_0 , are given by

$$W_i(v, u, x^k) = vW_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k), \quad (16)$$

$$H(v, u, x^k) = v^2\tilde{\sigma} + vH^{(1)}(u, x^k) + H^{(0)}(u, x^k), \quad (17)$$

$$\tilde{\sigma} \equiv \frac{1}{8} \left(4\sigma + W^{(1)i}W_i^{(1)} \right), \quad (18)$$

where σ is a constant. The remaining equations for CSI_0 that need to be solved are (hatted indices refer to an orthonormal frame in the transverse space):

$$W_{[\hat{i};\hat{j}]}^{(1)} = a_{\hat{i}\hat{j}}, \quad (19)$$

$$W_{(\hat{i};\hat{j})}^{(1)} - \frac{1}{2} \left(W_{\hat{i}}^{(1)} \right) \left(W_{\hat{j}}^{(1)} \right) = s_{\hat{i}\hat{j}}, \quad (20)$$

where the $a_{\hat{i}\hat{j}}$ and $s_{\hat{i}\hat{j}}$ and the components $R_{\hat{i}\hat{j}\hat{m}\hat{n}}^\perp$ are all constants (i.e., $dS_H^2 = g_{ij}^\perp(x^k)dx^i dx^j$ is curvature homogeneous). In four dimensions, $g_{ij}^\perp(x^k)dx^i dx^j$ is 2 dimensional, which immediately implies $g_{ij}^\perp(x^k)dx^i dx^j$ is a 2 dimensional locally homogeneous space and, in fact, maximally symmetric space. Up to scaling, there are (locally) only 3 such, namely the sphere, S^2 ; the flat plane, \mathbb{E}^2 ; and the hyperbolic plane, \mathbb{H}^2 .

The equations (19) and (20) now give a set of differential equations for $W_i^{(1)}$. These equations determine uniquely $W_i^{(1)}$ up to initial conditions (which may be free functions in u). Also, requiring CSI_1 gives an additional set of constraints:

$$\alpha_{\hat{i}} = \sigma W_{\hat{i}}^{(1)} - \frac{1}{2}(s_{\hat{j}\hat{i}} + a_{\hat{j}\hat{i}})W^{(1)\hat{j}}, \quad (21)$$

$$\beta_{\hat{i}\hat{j}\hat{k}} = W^{(1)\hat{n}}R_{\hat{n}\hat{i}\hat{j}\hat{k}}^\perp - W_{\hat{i}}^{(1)}a_{\hat{j}\hat{k}} + (s_{\hat{i}[\hat{j}} + a_{\hat{i}[\hat{j}})W_{\hat{k}]}^{(1)}, \quad (22)$$

where $\alpha_{\hat{i}}$ and $\beta_{\hat{i}\hat{j}\hat{k}}$ are constants determined from the curvature invariants. We note that for a four-dimensional Kundt spacetime, CSI_1 implies CSI [4].

There is a strong relationship between CSI spacetimes and those that are universal in four dimensions:

Theorem 3.1. *A 4D universal spacetime of Petrov type **D**, **II**, or **III**, is a Kundt-CSI spacetime.*

Proof. Consider type **D** first. Assuming that the spacetime is Einstein, then that spacetime is necessarily CSI_0 (which follows from the previous discussion). This implies that the boost weight 0 components are constants in the canonical frame. Using the Bianchi identities, it immediately follows that it is also Kundt. Since the previous analysis also implies that it is CSI_1 , then we have that the spacetime is Kundt-CSI.

For type **II**, the analysis is almost identical to the type **D** analysis. For type **III**, it is necessary to calculate some conserved tensors. Using the Weyl type **III** canonical form, the Bianchi identities imply that the Newman-Penrose spin coefficients [14] $\kappa = \sigma = 0$ (and $\rho = \epsilon$, $\beta = \tau$, $\alpha = -2\pi$, $\gamma = -2\mu$). Requiring also that $H_{ab}^{(6,3)(8)} = \lambda g_{ab}$, say, gives the additional equation: $\rho^2 = 0$. Clearly, $\rho = 0$ and the spacetime is Kundt. Since CSI_1 implies CSI for Kundt spacetimes, the theorem follows. \square

Although the theorem does not explicitly include Weyl type **N** and **I** spacetimes, it is believed that these are also Kundt. For type **I** the expressions for the conserved tensors are messy and unmanageable, and for type **N** it is necessary to compute a particular order 16 conserved tensor.

This proves the first two statements in the Conjecture 1.1, at least for Petrov types **D**, **II** and **III** in 4 dimensions. However, we can see that the last two statements are also true:

Proposition 3.2. *Consider a 4D Kundt CSI spacetime (M, g) . If (M, g) is universal, then Conjecture 1.1 is true.*

Proof. The proof utilises the results from [4]. Assuming an Einstein, Kundt-CSI spacetime, the cases reduces to those where the corresponding homogeneous spacetime (\tilde{M}, \tilde{g}) , is locally one of the following: Minkowski, de Sitter, anti-de Sitter, $dS_2 \times S^2$, or $AdS_2 \times H^2$. These have the corresponding Riemannian counterparts, (\hat{M}, \hat{g}) , with identical invariants: flat space, S^4 , H^4 , $S^2 \times S^2$, $H^2 \times H^2$. \square

We note that the opposite does not follow; namely, it is not true that for every Riemannian universal spacetime there is a Lorentzian spacetime with the same invariants. For example, the symmetric spaces CP^2 and $H_{\mathbb{C}}^2$ (with the corresponding Fubini-Study and Bargmann metrics, respectively) do not have Lorentzian counterparts. Thus the Conjecture 1.1 is signature dependent.

Finally, let us comment on the situation in higher dimensions, where much less is known. We have not yet proven that higher dimensional CSI spacetimes are either locally homogeneous or degenerate Kundt, which is necessary for a proof of a higher dimensional version of Proposition 3.2, although we have conjectured that this is so [3, 4]. It is also very likely (from the study of curvature operators) that universality implies that a spacetime is degenerate Kundt, but this has not been discussed explicitly to date. However, proving the higher dimensional version of the last part of Conjecture 1.1 will likely be more difficult since we first need the analytic extension of the Kundt spacetime.

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